Ljapunov Exponents, Hyperchaos and Hurst Exponent

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We consider nonlinear dynamical systems with chaotic and hyperchaotic behaviour. We investigate the behaviour of the Hurst exponent at the transition from chaos to hyperchaos. A two-dimensional coupled logistic map is studied.

Key words: Chaos; Hyperchaos; Hurst Exponent.

The application of fractal methods [1 – 6] allows us to explore the chaotic nature of time series. The methods used are (i) the time delay method [embedding dimension E, attractor (fractal) dimension D], (ii) Hurst's rescaled range analysis [Hurst exponent H, attractor (fractal) dimension D], (iii) Ljapunov exponents λ_j (entropy S), (iv) fractal dimensions (capacity, Hausdorff dimension, similarity dimension), (v) nonlinear prediction algorithm.

The Ljapunov exponents illustrate the bounded dynamical system's sensitivity on initial conditions [1 – 6]. Positive Ljapunov exponents of a bounded motion are considered as evidence of chaos, while negative exponents suggest a mean reverting behaviour. The larger the largest positive one-dimensional Ljapunov exponent, the more chaotic is the dynamical system and, conversely, the shorter is the time scale of the system's predictability. To characterize chaotic behaviour of a nonlinear dynamical system, the set of the one-dimensional Ljapunov exponents is used. If we have multiple positive one-dimensional Ljapunov exponents, we have hyperchaos. It requires at least a four-dimensional state space in the case of a first order autonomous differential equation. In two-dimensional maps we can also find hyperchaos. Often these dynamical systems depend on a bifurcation parameter, and we can study the transition from chaos to hyperchaos.

Einstein studied the properties of Brownian motion [7] (Wiener process) and found that the average distance $\sqrt{\langle(\Delta x)^2\rangle}$ covered by a particle undergoing random collisions is directly proportional to the square-root of the time Δt, *i. e.*,

$$\sqrt{\langle(\Delta x)^2\rangle} \propto (\Delta t)^{1/2}.$$

Hurst [8] (see also [4, 5, 9]) generalized Brownian motion for anomalous diffusion processes using

$$\sqrt{\langle(\Delta x)^2\rangle} \propto (\Delta t)^H$$

where H is the Hurst exponent. If $H = 1/2$, the behaviour of the time-series is similar to a random walk. If $H < 1/2$, the time series covers less "distance" than a random walk (*i.e.*, if the time-series increases, it is more probable that then it will decrease, and *vice versa*). Thus values of $0 < H < 0.5$ indicate anti-persistent behaviour. If $H > 1/2$, the time series covers more "distance" than a random walk (if the time series increases, it is more probable that it will continue to increase). Thus a value of $0.5 < H < 1$ indicates a so-called persistent behaviour (*i.e.*, one can expect with increasing certainty as the value moves towards 1 that whatever direction of change has been current will continue).

The rescaled range analysis R/S (range/standard derivation) developed by Hurst [8] provides a simple tool for analyzing the time series in the form of a so-called Hurst plot. The Hurst exponent H, which ranges between 0 and 1, can be derived as the slope in the Hurst plot, in which $\ln(R/S)$ is plotted against $\ln\tau$, where τ is the time step. Given a time series $x_t : t = 0, 1, \ldots, N-1$, the Hurst exponent can be estimated [5, 8, 9] by taking the slope (x/s) plotted versus n on a log-log scale. The rescaled range analysis method computes a ratio R/S defined as follows. The given time series

$$X := \{x_t : t = 0, 1, \ldots, N-1\}$$

is divided into ℓ intervals (boxes) of equal length n.

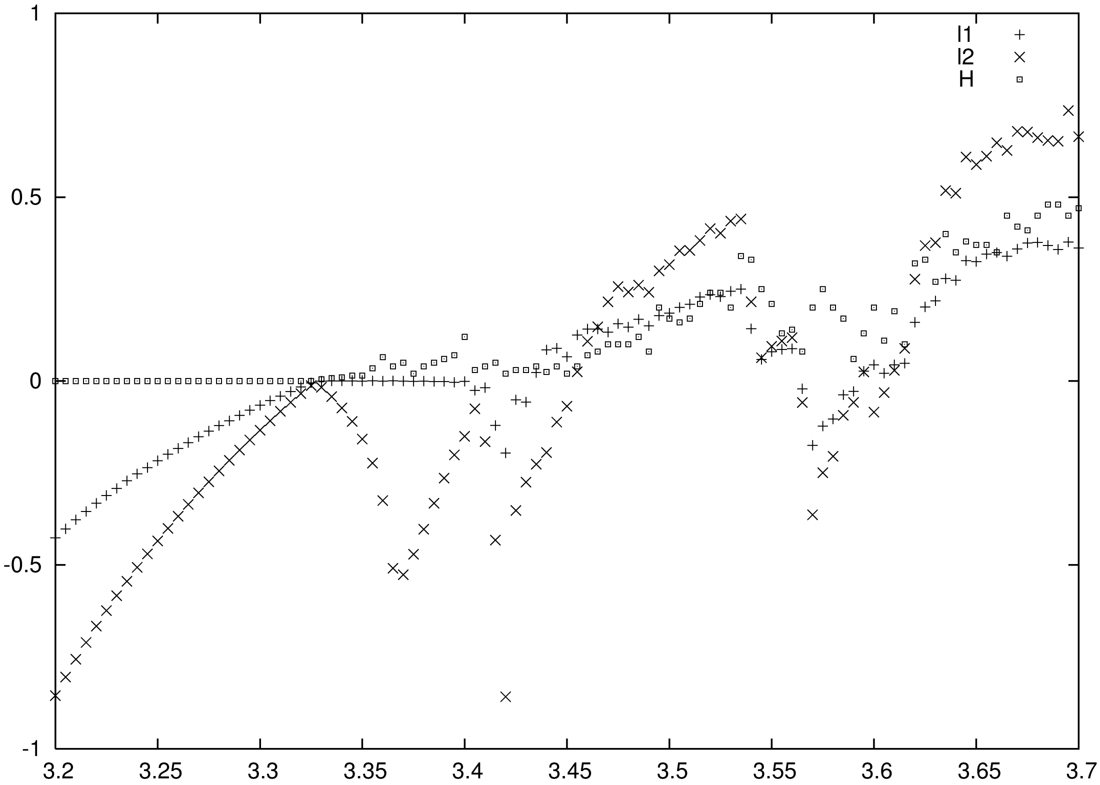

Fig. 1. Hurst and Ljapunov exponents as functions of r.

Thus $N = \ell \cdot n$. In the k^{th} box ($k = 0, 1, \ldots, \ell - 1$), there are n elements,

$$X_j^{(k)}(n) := \{x_j : j = k \cdot n, k \cdot n + 1, \ldots, k \cdot n - 1 + n\}.$$

Thus $x_j^{(k)} \equiv x_{k \cdot n + j}$. The local fluctuation at the point j in the k^{th} box, *i.e.* $(x_j^{(k)} - \langle x \rangle_n^{(k)})$ is calculated as the deviation from the mean

$$\langle x \rangle_n^{(k)} := \frac{1}{n} \sum_{j=0}^{n-1} x_j^{(k)}$$

in the k^{th} box. The cumulative departure $Y_m^{(k)}(n)$ up the m^{th} point in the k^{th} box (of size n) is calculated next

$$Y_m^{(k)}(n) := \sum_{j=0}^{m} (x_j^{(k)} - \langle x \rangle_n^{(k)}) = \left(\sum_{j=0}^{m} x_j^{(k)} \right) - (m+1) \langle x \rangle_n^{(k)}$$

for $m = 0, 1, \ldots, n - 1$ and in all k boxes. The rescaled range function is defined as

$$\frac{R^{(k)}}{S^{(k)}}(n) := \frac{\max_{0 \le m < n}(Y_m^{(k)}(n)) - \min_{0 \le m < n}(Y_m^{(k)}(n))}{\sqrt{(1/n) \sum_{j=0}^{n-1} \left(x_j^{(k)} - \langle x \rangle_n^{(k)} \right)^2}}$$

for $k = 0, 1, \ldots, \ell - 1$. The average of the rescaled range in all boxes with an equal size n is next obtained and denoted by $\langle R/S \rangle$. The above computation is then repeated for different values of n (and therefore ℓ) to provide a relationship between $\langle R/S \rangle$ and n. This is expected to be a power law $\langle R/S \rangle \approx n^H$ if some scaling range and law exist.

We study two dynamical systems which possess chaotic and hyperchaotic behaviour depending on a bifurcation parameter. We calculate the Hurst exponent and Ljapunov exponents for the chaotic and hyperchaotic domain.

As our example we consider the coupled logistic map

$$x_{1t+1} = r x_{1t}(1 - x_{1t}) + e(x_{2t} - x_{1t}),$$
$$x_{2t+1} = r x_{2t}(1 - x_{2t}) + e(x_{1t} - x_{2t}),$$

where $r \in [3, 3.7]$ and $e = 0.06$. Hogg and Hubermann [10] and Meyer-Kress and Haubs [11] have studied the regular, chaotic and hyperchaotic behaviour of this coupled map. Depending on the initial values and the parameter values one can find the following behaviour: (i) orbits tend to a fixed point, (ii) periodic behaviour, (iii) quasiperiodic behaviour, (iv) chaotic behaviour, (v) hyperchaotic behaviour and (vi) x_{1t} and x_{2t} explode, *i.e.*, for a finite time the values of x_{1t} and (or) x_{2t} tend to infinity. In the following we do not consider the last case by choosing the proper initial values. Before calculating the Ljapunov exponents, we run the program till the transients have been decayed. We use 1000 time steps for the decay of the transients. Figure 1 shows the largest one-dimensional λ_1^I, the two-dimensional Ljapunov exponent λ^{II} and the Hurst exponent H. The second one-dimensional Ljapunov exponent λ_2^I is calculated from $\lambda^{II} = \lambda_1^I + \lambda_2^I$. For the calculation of the Hurst exponent we use the length of the time series as $N = 16384$. To test the accuracy we also have used longer time series with $N = 32768$ and $N = 65536$. The Hurst exponent only changed by 0.005. For all values of r the Hurst exponent is in the range $\{0, 0.5\}$. To be anti-persistent is to tend to turn back toward the point one came from or, in terms of the random walk picture, to diffuse slower than in the ordinary Brownian motion. Any increasing trend in the past makes a decreasing trend in the future more probable, and *vice versa*. In the range $\{3.2, 3.325\}$ we have non-chaotic behaviour and the Hurst exponent is 0. In the range $\{3.325, 3.7\}$ we have non-chaotic, chaotic and hyperchaotic behaviour. We see that for the hyperchaotic case the Hurst exponent is closer to 0.5 than in the chaotic case. Thus the Hurst exponent $0 < H < 0.5$ means that there is no trend in the time series extracted from the system. This is not surprising since even in the hyperchaotic regime after removing the transients, the hyperchaotic orbit wanders in a chaotic attractor. Therefore, there is no possibility for persistence: an orbit point eventually comes closer to some small neighbourhood of any other orbit point. In other words, if the time-series increases, it is more probable that after a suitable time interval it will decrease, and *vice versa*, which corresponds precisely to the regime characterized by $0 < H < 0.5$.

H is related to the fractal dimension (capacity) D, via $H = 2 - D$. The numerical calculation of the capacity [4] for $r = 3.7$ provides $C \approx 1.55$. This agrees well with the Hurst exponent $H \approx 0.47$ and $C = 2 - H$. The numerical calculation of the capacity [4] is much more involved than that of the Hurst exponent. Note that if, for example $H = 0.1$, then the signal "fills in" siginificantly more space than, for example, $H = 0.9$ and consequently has a higher fractal dimension (in our case the capacity).

We also have a relation with the autocorrelation functions [4] $x_1(\tau)$ and $x_2(\tau)$. If we have a power law $x_j(\tau) \propto \tau^{-\alpha}$, then this is known as long-range memory or long-range dependence or persistence, where $H = 1 - \alpha/2$. The Hurst exponent H quantifies the strength of the persistence and consequently the degree of autocorrelation.

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